

Generalized FEM

1 Notation

This notation is based on the work of Nielsen (1987). Consider the problem of representing a field u over a region Ω by a finite number of piecewise functions. Note that the scheme presented here is not as general as the scheme proposed by Nielsen, but includes all of the fundamental concepts outlined there.

1. The region of interest Ω is approximated by Ω^E which is split up into a finite number of elements ω^ε . Note that Ω^E should approximate Ω as closely as possible, as the field representation will be defined only on Ω^E and its boundary.
2. Each element has L local coordinate directions, where the local coordinates (x^1, x^2, \dots, x^L) map the interior of the element into \mathbb{R}^L . It is frequently convenient, particularly when using Lagrange basis functions, to limit the local coordinates to range between 0 and 1.
3. Within each element ε , the field $u^{(\varepsilon)}(x)$ is defined by a finite number of linearly independent parameters $u^{\beta(\varepsilon)}$, called *element field parameters*.
4. The contribution of an element field parameter to the field at position x is given by an element basis function $\psi_{\beta(\varepsilon)}(x)$, $x \in \omega^\varepsilon$. The value of $\psi_{\beta(\varepsilon)}(x)$ represents the contribution that element field parameter $u^{\beta(\varepsilon)}$ makes to the field at x . Element basis functions are undefined outside the element in which their local coordinate system is defined.
5. At any point in the element, the field is given by $u^{(\varepsilon)}(x) = \psi_{\beta(\varepsilon)}(x) u^{\beta(\varepsilon)}$. Restricting $u^{(\varepsilon)}(x)$ to be a linear combination of the element field parameters enables the direct solution of many field approximation problems, since the governing equations will be linear.
6. The field approximation over the region Ω^ε is given by the union of the fields within each element. The resulting field is called the *ensemble field*, given by $U = \cup_\varepsilon u^{(\varepsilon)}(x)$.
7. In order to impose continuity conditions on the ensemble field, constraints are applied to the element field parameters $u^{\beta(\varepsilon)}$ in adjoining elements. The resulting equations will be linear in $u^{\beta(\varepsilon)}$ as the element field $u^{(\varepsilon)}(x)$ is a linear combination of $u^{\beta(\varepsilon)}$, with coefficients $\psi_{\beta(\varepsilon)}(x)$. The continuity conditions can be used to reduce the number of linearly independent parameters needed to describe the field.
8. The final set of linearly independent parameters used to define the ensemble field (after continuity conditions have been applied) are called the *ensemble field parameters* U^B .
9. There exists a real-valued matrix, $\nu_B^{\beta(\varepsilon)}$, which maps the ensemble field parameters to the element field parameters, $u^{\beta(\varepsilon)} = \nu_B^{\beta(\varepsilon)} U^B$. This is called the *ensemble to element parameter map*.
10. Each ensemble field parameter has an associated *ensemble basis function* given by $\Psi_B(x) = \cup_\varepsilon \psi_{\beta(\varepsilon)}(x) \nu_B^{\beta(\varepsilon)}$, where $X \in \Omega^\varepsilon$, $x \in \omega^\varepsilon$. The value of the ensemble basis function $\Psi_B(x)$ represents the contribution that the corresponding ensemble field parameter U^B makes to the field at X .

2 The Ensemble to Element Parameter Map

The ability to obtain the element field parameters from global parameters using a linear map allows the simple solution of extremely complex problems. The user may take any combination of basis functions in the same problem, and may choose the global parameters describing each field to be any parameters that can be expressed as a linear function of the element field parameters. Any constraints that can be expressed in the same way can be applied to the field. The map can even be used to create implicit coordinate transformations.

In the following sections the generation of the ensemble to element parameter map is discussed, and the process illustrated with a simple example.

2.1 Ensemble to Element Parameter Map Generation

Consider the problem of generating a matrix that maps the ensemble field parameters to the element field parameters. For simple problems, particularly those involving Lagrange elements and cubic Hermite elements, such a map can be generated by inspection. For more complicated basis functions, map generation is not so straightforward.

It is usually a simple procedure to express each ensemble field parameter in terms of the element field parameters — a linear relationship if the inverse is to be true. If the ensemble field parameter is a field value, then this will involve evaluating the basis functions in the element where the value is to come from, and multiplying by the element field parameters.

The number of degrees of freedom in the ensemble is reduced by enforcing constraints on the field. Again these must be formulated as linear functions of the element field parameters. Common constraints involve field and field derivative continuity between elements, which can be put together by evaluating the basis functions in neighbouring elements at inter-element boundaries, and equating the corresponding terms. The same method can also be used to set a field value or derivative at a specific location, or enforce constraints such as step changes in the field between elements.

The result of these first steps is N equations, comprised of three types:

1. m ensemble field parameter equations

$$U^B = a_{\beta(\varepsilon)} u^{\beta(\varepsilon)} \quad (1)$$

2. c_1 constraints with constant terms

$$c = a_{\beta(\varepsilon)} u^{\beta(\varepsilon)} \quad (2)$$

3. c_0 constraints with no constant term

$$0 = a_{\beta(\varepsilon)} u^{\beta(\varepsilon)} \quad (3)$$

It helps if all of these equations are linearly independent, a condition that can be harder to enforce than it appears, as will be shown later. The equations can be put together into a single system, where the mapping matrix is called the *element to ensemble parameter map*, not to be confused with the ensemble to element parameter map, which is the goal of this process:

$$\begin{pmatrix} U^B \\ c \\ 0 \end{pmatrix} = \begin{pmatrix} A_B \\ A_{C_1} \\ A_{C_0} \end{pmatrix} \begin{pmatrix} \vdots \\ u^{\beta(\varepsilon)} \\ \vdots \end{pmatrix} \quad (4)$$

If the model is properly constrained, then the total number of constraints is equal to the difference between the number of ensemble and element field parameters. If the constraints are linearly independent, then the element to ensemble parameter map can be inverted. Setting $\nu^{\beta(\varepsilon)} = A^{-1}$, and moving it to the other side

$$\begin{aligned}
\begin{pmatrix} \vdots \\ u^{\beta(\varepsilon)} \\ \vdots \end{pmatrix} &= \begin{pmatrix} \vdots \\ \nu^{\beta(\varepsilon)} \\ \vdots \end{pmatrix} \begin{pmatrix} U^B \\ c \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} \vdots \\ \nu_B^{\beta(\varepsilon)} & \nu_{C_1}^{\beta(\varepsilon)} & \nu_{C_0}^{\beta(\varepsilon)} \\ \vdots \end{pmatrix} \begin{pmatrix} U^B \\ c \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} \vdots \\ \nu_B^{\beta(\varepsilon)} \\ \vdots \end{pmatrix} U^B + \begin{pmatrix} \vdots \\ u_c^{\beta(\varepsilon)} \\ \vdots \end{pmatrix} \tag{5}
\end{aligned}$$

where $u_c^{\beta(\varepsilon)}$ is the sum of the columns in $\nu_{C_1}^{\beta(\varepsilon)}$ multiplied by the constants in c . Note that $\nu_{C_0}^{\beta(\varepsilon)}$ has no impact on the solution process, and so can be discarded. For the majority of FE problems, there are no constraints with constant terms, and so $\nu_{C_1}^{\beta(\varepsilon)}$ and $u_c^{\beta(\varepsilon)}$ will not be present in further examples and discussion.

If the first m rows of A correspond to ensemble field parameter equations, the ensemble to element parameter map $\nu_B^{\beta(\varepsilon)}$ will consist of the first m columns of $\nu^{\beta(\varepsilon)}$. Because parts of the inverse may not be needed, it is not efficient to calculate it explicitly. In this study, A was LU-factorised, and forward- and backward-substitution used to solve for the necessary number of columns.

For problems with multi-dimensional elements, it may be extremely difficult to find the right number of linearly independent constraints, where many more constraints can be formulated than are necessary, but where many are interdependent. Instead of trying to reduce the number of constraints using techniques such as Gaussian elimination, it is possible to apply a least squares approach where, if the constraints are consistent, the least squares error will have value zero.

Instead of solving

$$\begin{pmatrix} U^B \\ c \\ 0 \end{pmatrix} = \begin{pmatrix} \vdots \\ A \\ \vdots \end{pmatrix} \begin{pmatrix} \vdots \\ u^{\beta(\varepsilon)} \\ \vdots \end{pmatrix} \tag{6}$$

the normal equations are solved

$$A^T \begin{pmatrix} U^B \\ c \\ 0 \end{pmatrix} = A^T A u^{\beta(\varepsilon)} \tag{7}$$

which gives

$$\nu^{\beta(\varepsilon)} = (A^T A)^{-1} A^T \tag{8}$$

The ensemble to element parameter map $\nu_B^{\beta(\varepsilon)}$ is again made up of the first m columns of $\nu^{\beta(\varepsilon)}$. If $A^T A$ is LU-factorised, then the i^{th} column of $\nu^{\beta(\varepsilon)}$, can be found by forward- and backward-substitution, using the i^{th} row of A as the right hand side. The consistency of the constraints by checking that $A \cdot \nu^{\beta(\varepsilon)} = I$ (the identity). Even if the constraints are inconsistent, this method can be used to provide a compromise between conflicting constraints.

Another problem occurs when the number of constraints that can be formulated is not enough to fully specify the behaviour of the field within each element. If this happens, then A is not full rank, and can not be inverted. Furthermore an infinite number of solutions (maps) exist. One common method for solving rank-deficient least squares problems is singular value decomposition (SVD). The A matrix can be factorised into the product of two orthogonal matrices P and Q and one diagonal matrix S , $A = PSQ$. The map can then be evaluated by solving for columns of the identity.

$$P S Q \nu_B^{\beta(\varepsilon)} = I_B \quad (I_B \text{ is the } B^{\text{th}} \text{ column of the identity}) \quad (9)$$

Solution using SVD decomposition of a rank deficient matrix produces vectors with the minimum 2-norm which satisfy the constraints (Golub & Van Loan 1996). In terms of ensemble to element parameter map generation, this means that the resulting map will be fairly even (in terms of magnitude) in areas where insufficient constraints exist. A disadvantage associated with this property of SVD factorisation is that the field may depend on ensemble field parameters in non-adjacent elements in areas of the model where the field is not fully constrained. This leads to a loss of sparsity in the map and has significant implications for the speed of the data fitting process.

2.2 A Simple Example

In this section, the formulation and use of an ensemble to element parameter map is demonstrated for the simple three element model shown. In order to show the power of the mapping technique an odd and somewhat unlikely combination of basis functions and continuity conditions are used to define the field.

In the first element, the field is defined by linear Lagrange basis functions. This requires two parameters $u^{1(1)}$ and $u^{2(1)}$, with corresponding basis functions

$$\psi_{1(1)}(x) = 1 - x \quad \psi_{2(1)}(x) = x$$

A cubic Bézier basis function will be used to define the field in the second element, with four parameters $u^{1(2)} - u^{4(2)}$, and corresponding basis functions

$$\begin{aligned} \psi_{1(2)}(x) &= 1 - 3x + 3x^2 - x^3 \\ \psi_{2(2)}(x) &= 3x - 6x^2 + 3x^3 \\ \psi_{3(2)}(x) &= 3x^2 - 3x^3 \\ \psi_{4(2)}(x) &= x^3 \end{aligned}$$

The field in the third element will be defined by a quadratic monomial basis function, with three parameters $u^{1(3)} - u^{3(3)}$, and corresponding basis functions

$$\psi_{1(3)}(x) = 1 \quad \psi_{2(3)}(x) = x \quad \psi_{3(3)}(x) = x^2$$

Certain continuity constraints on the field are to be enforced at the inter-element boundaries. Between elements one and two, the element should be C^1 continuous, that is the field and field

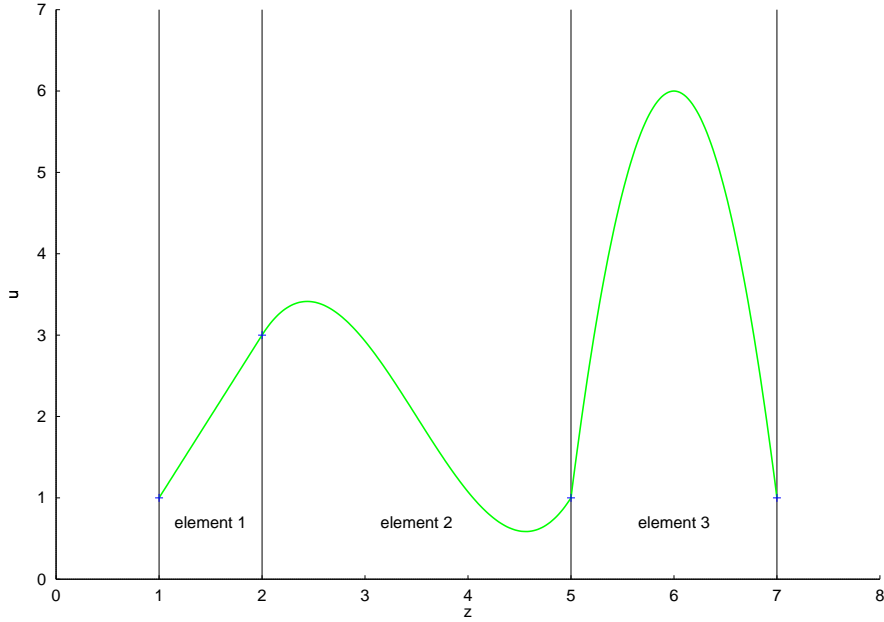


Figure 1: A three element problem, constrained by an ensemble to element parameter map. The first element, ε_1 is linear Lagrange, the second element cubic Bézier, and the third element quadratic monomial.

derivative values at $x = 1$ in element one and at $x = 0$ in element two should be equal. Furthermore, it is required that the field is continuous across the boundary between elements two and three.

With nine element field parameters and three constraints in the problem, a total of six free ensemble parameters are required to define the field. These parameters can be chosen to be any convenient aspects of the field, but generally values that are known or of interest are picked. Let the first four parameters be the field values at $z = 1$, $z = 5$, $z = 6$ and $z = 7$, and the last two the field derivatives (with respect to global coordinate z) at $z = 2$ and $z = 5$ (for the field in element two).

Consider now how to express the ensemble parameters as linear combinations of the element field parameters:

1. It was previously decided that the first ensemble field parameter should be the value of the field at $z = 1$. This is in element one at $x = 0$. The field there is given by

$$\begin{aligned} u^{(1)}(0) &= \psi_{1(1)}(0) u^{1(1)} + \psi_{2(1)}(0) u^{2(1)} \\ &= (1 - 0) u^{1(1)} + (0) u^{2(1)} \\ &= u^{1(1)} \end{aligned}$$

Therefore $U^1 = u^{1(1)}$.

2. The field at $z = 5$ corresponds to $x = 0$ in element three. One could equally well have used $x = 1$ in element two, but that would yield a more complicated expression. Evaluating the field at the chosen point

$$U^2 = u^{(3)}(0) = u^{1(3)}$$

3. The field at $z = 6$ corresponds to $x = \frac{1}{2}$ in element three. Here

$$U^3 = u^{(3)} \left(\frac{1}{2} \right) = u^{1(3)} + \frac{1}{2} u^{2(3)} + \frac{1}{4} u^{3(3)}$$

4. $z = 7$ corresponds to $x = 1$ in element three. Here

$$U^4 = u^{(3)} (1) = u^{1(3)} + u^{2(3)} + u^{3(3)}$$

5. When evaluating the field derivative at $z = 2$, it is possible to use the field in element one or two. Element one, which is linear, will be simpler to deal with, so $x = 1$ in element one is considered.

$$\begin{aligned} \left. \frac{\partial u}{\partial z} \right|_{z=2} &= \frac{\partial u^{(1)} (1)}{\partial x} \frac{\partial x}{\partial z} && \left(\text{note that } \frac{\partial x}{\partial z} = 1 \right) \\ &= \frac{\partial \psi_{1(1)} (1)}{\partial x} u^{1(1)} + \frac{\partial \psi_{2(1)} (1)}{\partial x} u^{2(1)} \\ U^5 &= -u^{1(1)} + u^{2(1)} \end{aligned}$$

6. Finally the field derivative at $z = 5$ in element two is evaluated. Note that, in this element, the local coordinates must be scaled to match the global coordinate system.

$$\begin{aligned} \left. \frac{\partial u}{\partial z} \right|_{z=5} &= \frac{\partial u^{(2)} (1)}{\partial x} \frac{\partial x}{\partial z} && \left(\text{this time } \frac{\partial x}{\partial z} = \frac{1}{3} \right) \\ &= \frac{1}{3} \sum_{i=1 \dots 4} \left(\frac{\partial \psi_{i(2)} (1)}{\partial x} u^{i(2)} \right) \\ U^6 &= -u^{3(2)} + u^{4(2)} \end{aligned}$$

Now the three constraints that are to be applied to the field must be formulated as linear constraints on the element field parameters of neighbouring elements.

1. To enforce field continuity between elements one and two, the field at $x = 1$ in element one must equal the field at $x = 0$ in element two.

$$\begin{aligned} u^{(1)} (1) &= u^{(2)} (0) \\ \sum_{i=1 \dots 2} (\psi_{i(1)} (1) u^{i(1)}) &= \sum_{i=1 \dots 4} (\psi_{i(2)} (0) u^{i(2)}) \\ u^{2(1)} &= u^{1(2)} \end{aligned}$$

The resulting constraint is

$$u^{2(1)} - u^{1(2)} = 0$$

2. To force field derivative continuity between elements one and two, the same approach is applied to the field derivatives, within the global coordinate system.

$$\begin{aligned} \frac{\partial u^{(1)} (1)}{\partial z} &= \frac{\partial u^{(2)} (0)}{\partial z} \\ \left. \frac{\partial x}{\partial z} \right|_{\varepsilon_1} \sum_{i=1 \dots 2} \left(\frac{\partial \psi_{i(1)} (1)}{\partial x} u^{i(1)} \right) &= \left. \frac{\partial x}{\partial z} \right|_{\varepsilon_2} \sum_{i=1 \dots 4} \left(\frac{\partial \psi_{i(2)} (0)}{\partial x} u^{i(2)} \right) \\ -u^{1(1)} + u^{2(1)} &= \frac{1}{3} (-3u^{1(2)} + 3u^{2(2)}) \end{aligned}$$

The resulting constraint is

$$-u^{1(1)} + u^{2(1)} + u^{1(2)} - u^{2(2)} = 0$$

3. The constraint arising from enforcing field continuity between elements two and three is

$$u^{4(2)} - u^{1(3)} = 0$$

The element to ensemble parameter map can now be assembled.

$$\begin{bmatrix} U^1 \\ U^2 \\ U^3 \\ U^4 \\ U^5 \\ U^6 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} u^{1(1)} \\ u^{2(1)} \\ u^{1(2)} \\ u^{2(2)} \\ u^{3(2)} \\ u^{4(2)} \\ u^{1(3)} \\ u^{2(3)} \\ u^{3(3)} \end{bmatrix}$$

With six ensemble parameters and three zero constraints, the ensemble to element parameter map is the first six columns of the inverse of the element to ensemble parameter map.

$$\begin{bmatrix} u^{1(1)} \\ u^{2(1)} \\ u^{1(2)} \\ u^{2(2)} \\ u^{3(2)} \\ u^{4(2)} \\ u^{1(3)} \\ u^{2(3)} \\ u^{3(3)} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -3 & 4 & -1 & 0 & 0 \\ 0 & 2 & -4 & 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} U^1 \\ U^2 \\ U^3 \\ U^4 \\ U^5 \\ U^6 \end{bmatrix}$$

The field in was generated with the following set of ensemble field parameters.

$$U^B = [1 \quad 1 \quad 6 \quad 1 \quad 2 \quad 2]^T$$

Using the ensemble to element parameter map, it is possible to generate ensemble basis functions for each of the ensemble field parameters, using

$$\Psi_B(x) = \cup_{\varepsilon} \psi_{\beta(\varepsilon)}(x) \nu_B^{\beta(\varepsilon)}$$

For example, the ensemble basis function for the second ensemble parameter, the field at

the boundary between elements two and three, can be calculated in the following manner

$$\begin{aligned} \Psi_2(X) &= \cup_{\epsilon} \psi_{\beta(\epsilon)}(x) \nu_2^{\beta(\epsilon)} \\ &= \begin{bmatrix} (1-x)_{\epsilon_1} \\ (x)_{\epsilon_1} \\ (1-3x+3x^2-x^3)_{\epsilon_2} \\ (3x-6x^2+3x^3)_{\epsilon_2} \\ (3x^2-3x^3)_{\epsilon_2} \\ (x^3)_{\epsilon_2} \\ (1)_{\epsilon_3} \\ (x)_{\epsilon_3} \\ (x^2)_{\epsilon_3} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ -3 \\ 2 \end{bmatrix} \\ &= (3x^2 - x^3)_{\epsilon_2} + (1 - 3x + 2x^2)_{\epsilon_3} \end{aligned}$$

The form of this and the other ensemble basis functions is shown below. Although they are rarely useful in practical problems, they provide a useful visual guide to the support of each ensemble field parameter.

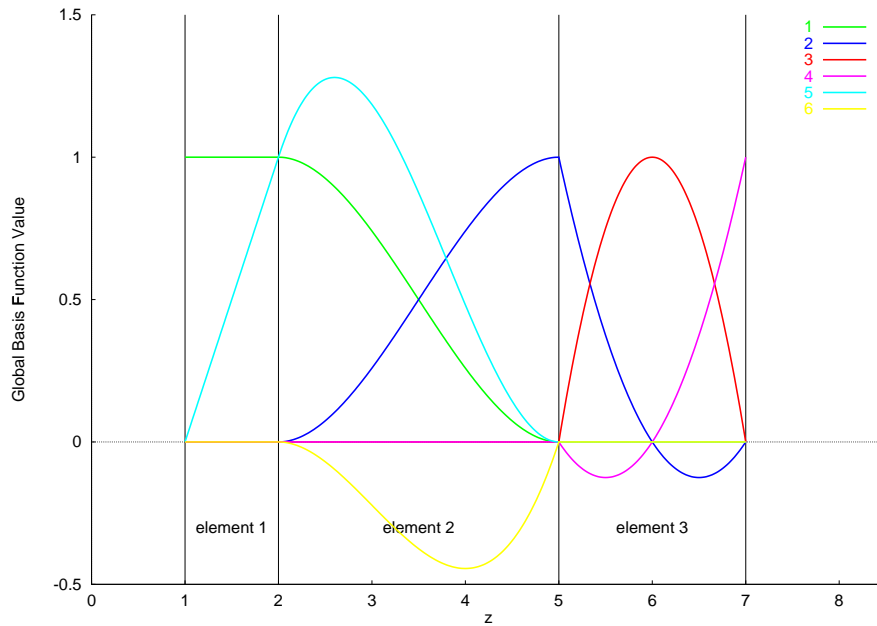


Figure 2: The global basis functions corresponding to each of the ensemble field parameters in the three element example.